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Exercise (Differential Equations)

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Part A: Key Concepts

1. **Definition** (Differential Equations). A **differential equation** is an equation containing derivatives as an unknown function. The **order** of a differential equation is the order of the highest derivative in the equation.

   Example.

Differential Equation	Order	Unknown Function
$\frac{dy}{dx} = 4y$	1	$y(x)$
$y'' + 2y = 2x$	2	$y(x)$
$\frac{d^3y}{dt^3} - t\frac{dy}{dt} + t(y-1) = e^t$	3	$y(t)$

   Example.

   (a) $y' = x$.

   To solve $y' = x$, we integrate the equation with respect to x once,

   $$y = \int x dx = \frac{1}{2}x^2 + C,$$

   where $C \in \mathbb{R}$. This is called the **general solution**.

   If the integration constant C is set to be a particular number, e.g. $C = 3$, then $y = \frac{1}{2}x^2 + 3$ is a **particular solution** to the differential equation.

   (b) The general solution to a differential equation contains an arbitrary constant. To determine a unique solution, some conditions need to be described. For example, consider

   $$\begin{cases} y' = x, \\ y(0) = 3. \end{cases}$$

   The general solutions are $y = \frac{1}{2}x^2 + C$. By using the **initial condition** $y(0) = 3$, we get $C = 3$. This problem is called an **initial condition problem**.

2. **Definition** (Separable Equation). Equations in the following form

   $$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

   are called **separable**.

   To solve the above equation, we rewrite it as

   $$h(y)\frac{dy}{dx} = g(x)$$

   Integrate with respect to x,

   $$\int h(y)dy = \int g(x)dx,$$

$$H(y) = G(x) + C,$$

which determines the function (relation) y about x.

It is more convenient to denote the equation by

$$h(y)dy = g(x)dx,$$

and the name **"separable"** just means the variables x and y are separated on two sides.

Example. Solve the differential equation

$$\frac{dy}{dx} = \frac{2x^3}{y^2}.$$

Then solve the initial value problem

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{2x^3}{y^2}, \\ y(0) = 1. \end{cases}$$

Solution. The equation is rewritten as

$$y^2 dy = 2x^3 dx.$$

Integrate both sides,

$$\int y^2 dy = \int 2x^3 dx,$$

Hence the general solution is

$$\frac{1}{3}y^3 = \frac{1}{2}x^4 + C.$$

Substituting the initial condition $y(0) = 1$ into the general solution yields $C = \frac{1}{3}$. Thus, the solution to the initial value problem is

$$\frac{1}{3}y^3 = \frac{1}{2}x^4 + \frac{1}{3}.$$

3. **Definition** (First-Order Linear Differential Equation). Equations in the following form

$$\frac{dy}{dx} + p(x)y = q(x)$$

are called **first-order linear differential equations**.

If $p(x) = 0$, integrate the equation $\frac{dy}{dx} = q(x)$ to get

$$y = \int q(x)dx.$$

If $p(x) \neq 0$, multiply the equation by some function $\mu = \mu(x)$,

$$\mu\frac{dy}{dx} + \mu p(x)y = \mu q(x). \tag{1}$$

This can be done as long as μ satisfies

$$\frac{d\mu}{dx} = \mu p(x),$$

which is a separable equation so that the left-hand side is a derivative of μy, that is,

$$\mu\frac{dy}{dx} + \mu p(x)y = \frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y.$$

This can be done as long as μ satisfies

$$\frac{d\mu}{dx} = \mu p(x),$$

which is a separable equation whose general solution is given by

$$\mu = e^{\int p(x)dx + \tilde{C}}. \tag{2}$$

(Note: μ is called the *integrating factor* and plays an important role in the general solution of the original first-order differential equation as shown below. It can be proved that the constant $\tilde{C}$ here will not affect the final general solution of the original differential equation. Therefore, it is common to directly set $\tilde{C} = 0$ and take $\mu = e^{\int p(x)dx}$.)

Returning to (1), we have

$$\frac{d}{dx}(\mu y) = \mu q(x)$$

Hence

$$\mu y = \int \mu q(x)dx + C$$

$$y = \frac{1}{\mu}(\int \mu q(x)dx + C) \tag{3}$$

Example. Solve

$$\frac{dy}{dx} - y = e^{3x}.$$

Solution. It is a first order linear equation with $p(x) = -1$ and $q(x) = e^{3x}$. Let

$$\mu = e^{\int p(x)dx} = e^{-x}.$$

(Caution: it should be e^{-x+C}, but we choose $C = 0$.)

Multiply the equation by μ,

$$\frac{d}{dx}(e^{-x}y) = e^{-x}e^{3x} = e^{2x}.$$

So

$$e^{-x}y = \int e^{2x}dx = \frac{1}{2}e^{2x} + C$$

(Caution: this C cannot be omitted.)

Hence, the general solution is

$$y = e^{x}(\frac{1}{2}e^{2x} + C).$$

4. **Definition** (Second-Order Differential Equations with Constant Coefficients)

We consider the following **second-order differential equation with constant coefficients**

$$ay'' + by' + cy = 0,$$

where a, b, c are constants, and $a \neq 0$.

Let's try exponential functions $y = e^{rx}$, then $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Substitute them into the equation,

$$(ar^2 + br + c)e^{rx} = 0.$$

Since e^{rx} is always nonzero, we have

$$ar^2 + br + c = 0$$

, which is called the **characteristic equation** of the differential equation. There are three cases to the roots of the characteristic equation, depending on the discriminant $\Delta = b^2 - 4ac$.

Case I: $b^2 - 4ac > 0$.

We have two distinct real roots $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. We can then show that both $y = e^{r_1x}$ and $y = e^{r_2x}$ satisfy the given differential equation. Consequently, the general solution is

$$y = C_1e^{r_1x} + C_2e^{r_2x}.$$

Example. Find the general solution to

$$y'' - y' - 6y = 0.$$

Solution. The characteristic equation is

$$r^2 - r - 6 = 0.$$

So the roots are $r_1 = 3$ and $r_2 = -2$. Thus, the general solution is

$$y = C_1 e^{3x} + C_2 e^{-2x}.$$

Case II: $b^2 - 4ac = 0$.

One repeated real root $r = -\dfrac{b}{2a}$. We have one solution $y_1 = e^{rx}$, and it can be checked that $y_2 = xe^{rx}$ is another solution. Hence, the general solution is

$$y = C_1 e^{rx} + C_2 x e^{rx}.$$

Case III: $b^2 - 4ac < 0$

Since the derivation involves complex numbers, we will present the general solution directly here

$$y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x,$$

where $\alpha = -\dfrac{b}{2a}$ and $\beta = \dfrac{\sqrt{4ac - b^2}}{2a}$.

Example. Find the general solution to

$$y'' - 4y' + 5y = 0.$$

Solution. The characteristic equation is

$$r^2 - 4r + 5 = 0.$$

$\alpha = 2$ and $\beta = 1$ gives the general solution

$$y = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x.$$

Summary of general solutions to $ay'' + by' + cy = 0$:

Discriminant	Nature of roots	General solutions
$b^2 - 4ac > 0$	$r_1 \neq r_2$, real	$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
$b^2 - 4ac = 0$	$r_1 = r_2$, real	$y = C_1 e^{r_1 x} + C_2 x e^{r_1 x}$
$b^2 - 4ac < 0$	$r_1 = \alpha + i\beta,\ r_2 = \alpha - i\beta$, complex	$y = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$

Part B: Basic Questions

1. Solve the following separable ODEs.

   (a) $\dfrac{dy}{dx} = -4x^2y^3$;

   (b) $\dfrac{dy}{dx} = \dfrac{y^2+1}{xy}$ with $y(1) = 3$;

   (c) $\dfrac{dy}{dx} = xe^{y-x^2}$ with $y(1) = 0$.

2. Solve the following first-order linear ODEs.

   (a) $y' + y = xe^{-x} + 1$;

   (b) $ty' + 2y = 2\sin t,\ y(\frac{\pi}{2}) = 1,\ t > 0$.

3. Find the solution of the initial value problem

$$2y'' - 3y' + y = 0,\ y(0) = 2,\ y'(0) = \frac{1}{2}.$$

Part C: Advanced Questions

4. (Motion under gravity and air resistance) Consider a ball with mass $m = 0.5$ kg with initial velocity $v_0 = 9.8$ m/s. Let $v(t)$ be the velocity at time t, gravity $g = 9.8$ m/s^2, and air resistance be $-\frac{v}{2}$. Find the formula for the velocity $v(t)$.

5. A 1000 liter tank is initially filled with a sugar solution with a concentration of 30 g/L. Another sugar solution with a concentration of 5 g/L flows into the tank at a rate of 10 L/min, and the well-mixed solution is drained out of the tank at a rate of 15 L/min.

   (a) At what time will the tank be half full?

   (b) When the tank is half full, how many grams of sugar are contained in the tank?

6. Consider the first-order linear differential equation

$$P(x)y' + Q(x)y = 0,$$

where $P(x)$, $Q(x)$ are given functions. Prove that if $y_1(x)$ and $y_2(x)$ are two solutions to the above equation, then for any constants c_1 and c_2, the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution.

7. Consider the second-order linear equation

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

where $P(x)$, $Q(x)$, $R(x)$ are given functions. Show that if $y_1(x)$ and $y_2(x)$ are two solutions to the above equation, then for any constants c_1 and c_2, the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution.

8. A free vibration is governed by

$$\begin{cases} y''(t) + 4y = 0, \\ y(0) = 1,\ y'(0) = 4, \end{cases}$$

Where $y(t)$ is the position at t. Find the period and frequency of the vibration.

Part D: Solutions

1. (a) If $y = 0$, it is easy to see that the differential equation holds.

   If $y \neq 0$, we have

$$\frac{1}{y^3}dy = -4x^2dx$$

   By integrating both sides, we have

$$\begin{aligned}
\int \frac{1}{y^3}dy &= \int -4x^2dx \\
\frac{1}{-2y^2} &= \frac{-4x^3}{3} + C \\
\frac{1}{y^2} &= \frac{8x^3 + C}{3} \\
y &= \pm\sqrt{\frac{3}{8x^3 + C}}
\end{aligned}$$

   Therefore, the solutions are $y = 0$ or $y = \pm\sqrt{\dfrac{3}{8x^3 + C}}$, where C is a a constant.

   (b) From the given differential equation, we have

$$\frac{y}{y^2+1}dy = \frac{1}{x}dx$$

   By integrating both sides, we have

$$\begin{aligned}
\int \frac{y}{y^2+1}dy &= \int \frac{1}{x}dx \\
\int \frac{1}{2} \cdot \frac{2y}{y^2+1}dy &= \int \frac{1}{x}dx \\
\frac{1}{2}\int \frac{1}{y^2+1}d(y^2) &= \int \frac{1}{x}dx \\
\frac{1}{2}\ln|y^2+1| &= \ln|x| + C \\
\ln(y^2+1) &= \ln(x^2) + C \\
y^2 + 1 &= e^{\ln(x^2)+C} \\
y &= \pm\sqrt{e^C x^2 - 1}
\end{aligned}$$

   Using the given condition $y(1) = 3$, we have

$$9 = e^C - 1 \Rightarrow e^C = 10$$

   Therefore, the solution is

$$y = \sqrt{10x^2 - 1}.$$

   (c) From the given differential equation, we have

$$e^{-y}dy = xe^{-x^2}dx$$

   By integrating both sides,

$$\begin{aligned}
\int e^{-y}dy &= \int xe^{-x^2}dx \\
\int -e^{-y}dy &= \int -xe^{-x^2}dx \\
\int -e^{-y}dy &= \frac{1}{2}\int (-2x)e^{-x^2}dx \\
e^{-y} &= \frac{1}{2}e^{-x^2} + C
\end{aligned}$$

Since $y(1)=0$, we have $e^{-0}=\frac{1}{2}e^{-1^2}+C \Rightarrow C=1-\dfrac{1}{2e}$.

Hence, we have

$$e^{-y}=\frac{1}{2}e^{-x^2}+1-\frac{1}{2e}$$
$$y=-\ln\left(\frac{1}{2}e^{-x^2}+1-\frac{1}{2e}\right)$$

2. (a) The integrating factor is $\mu(x)=e^{\int 1dx}=e^x$. Therefore, we have

$$e^x(y'+y)=e^x(xe^{-x}+1)$$
$$(e^xy)'=x+e^x$$
$$e^xy=\frac{x^2}{2}+e^x+C$$
$$y=\frac{x^2}{2}e^{-x}+1+Ce^{-x}$$

(b) We have $\mu(x)=e^{\int \frac{2}{t}dt}=t^2$ and so

$$\frac{d}{dt}(t^2y)=2t\sin t$$
$$t^2y=\int 2t\sin t dt=-2t\cos t+2\sin t+C$$

Since $y(\frac{\pi}{2})=1$, we have $C=\dfrac{\pi^2}{4}-2$, and the solution is $y=t^{-2}(-2t\cos t+2\sin t+\dfrac{\pi^2}{4}-2)$.

3. The characteristic equation is given by $2r^2-3r+1=0$, hence $r_1=1$ and $r_2=\dfrac{1}{2}$.

Therefore, the general solution is $y=C_1e^t+C_2e^{\frac{t}{2}}$.

$y(0)=2$ gives $C_1+C_2=2$.

$y'(0)=\dfrac{1}{2}$ gives $C_1+\dfrac{C_2}{2}=\dfrac{1}{2}$.

Solving the above equations, we get $C_1=-1$ and $C_2=3$.

Therefore, the solution is $y=-e^t+3e^{\frac{t}{2}}$

4. Select the upward direction as the positive direction for velocity and force. According to Newton's Second Law, we have

$$F(t)=ma(t)=mv'(t)=-mg-\frac{v(t)}{2},$$

Hence $v'(t)+v(t)=-9.8$, with the initial condition $v(0)=9.8$, we solve the first order linear equation to obtain

$$v(t)=-9.8+19.6e^{-t}.$$

5. (a) Let t represent the time, then the volume of solution in the tank at time t is $V(t)=1000-15t+10t$. We can solve $500=1000-15t+10t$ and get $t=100$.

(b) Let $S(t)$ be the amount of sugar at time t.

$$\begin{cases} S(0)=1000\times 30 \\ S'(t)=\text{rate in}-\text{rate out}=5\times 10-15\times\dfrac{S(t)}{V(t)}=50-\dfrac{3S(t)}{200-t}, \quad 0\le t<200 \end{cases}$$

which is a first-order linear ODE. Let $\mu(x)=e^{\int\frac{3}{200-t}dt}=(200-t)^{-3}$, multiplied to both sides of the ODE, we get

$$[(200-t)^{-3}S]'=50(200-t)^{-3}$$
$$(200-t)^{-3}S=50\int(200-t)^{-3}dt=25(200-t)^{-2}+C$$

$$S = 25(200 - t) + C(200 - t)^3$$

By the initial condition $S(0) = 30000$, we have $C = \dfrac{1}{320}$.

So the solution $S(t) = 25(200 - t) + \dfrac{(200 - t)^3}{320}$.

And $S(100) = 5625$.

6. Substituting $y = c_1 y_1(x) + c_2 y_2(x)$ into the equation, we have

$$\begin{aligned}
&P(x)y' + Q(x)y \\
&= P(x)(c_1y_1 + c_2y_2)' + Q(x)(c_1y_1 + c_2y_2) \\
&= P(x)(c_1y_1' + c_2y_2') + Q(x)(c_1y_1 + c_2y_2) \\
&= c_1(P(x)y_1' + Q(x)y_1) + c_2(P(x)y_2' + Q(x)y_2) \\
&= 0
\end{aligned}$$

Therefore, $y = c_1 y_1(x) + c_2 y_2(x)$ is also a solution.

7. Substituting $y = c_1 y_1(x) + c_2 y_2(x)$ into the equation, we have

$$\begin{aligned}
&P(x)y'' + Q(x)y' + R(x)y \\
&= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\
&= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\
&= c_1(P(x)y_1'' + Q(x)y_1' + R(x)y_1) + c_2(P(x)y_2'' + Q(x)y_2' + R(x)y_2) \\
&= 0
\end{aligned}$$

Therefore, $y = c_1 y_1(x) + c_2 y_2(x)$ is also a solution.

8. The solution is $y = 2\sin(2t) + \cos(2t)$. Hence, the period is π and the frequency is $\dfrac{1}{\pi}$.